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Solution

$$f(t) = t + u(t-3)(\cos t - t) + u(t-6)(7 - \cos t)$$

The Laplace transform of $u(t-a)$ is e^{-as}/s .
 If f is a function with Laplace transform $F(s)$, then
 $L\{u(t-a)f(t-a)\} = e^{-as}F(s)$.

$$\begin{aligned} \text{Then } L(f(t)) &= L\left(t + u(t-3)(\cos t - t) + u(t-6)(7 - \cos t)\right) \\ &= L(t) + L(u(t-3)(\cos t - t)) + L(u(t-6)(7 - \cos t)) \\ &= \frac{1}{s^2} + e^{-3s}L(\cos t - t) + e^{-6s}L(7 - \cos t) \end{aligned}$$

$$\text{Consider } e^{-3s}L(\cos t - t) = \int_3^6 e^{-st}(\cos t - t)dt \times e^{3s}$$

$$\begin{aligned} \text{from } L\{g(t)u(t-a)\} &= \int_0^{\infty} e^{-st}g(t)u(t-a)dt \\ &= \int_a^{\infty} e^{-st}g(t)dt \end{aligned}$$

$$\begin{aligned} \text{Therefore } e^{-3s}L(\cos t - t) &= \int_3^6 e^{-3s} \cdot e^{-st}(\cos t - t)dt \\ &= e^{-3s} \int_3^6 e^{-st}(\cos t - t)dt \end{aligned}$$

$$\text{Then } e^{-3s} \left[\int_3^6 e^{-st} \cos t dt - \int_3^6 t e^{-st} dt \right] \dots$$

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Using integration by parts.

let $M(t) = \int_3^6 e^{-st} \cosh t \, dt$ let $u = e^{-st}$ $du = -s e^{-st} dt$
 $dv = \cosh t \, dt$ $v = \sinh t$ ($\int dv = \int \cosh t \, dt$)

$$\int u \frac{dv}{dt} dt = uv - \int v \frac{du}{dt} dt.$$

$$\Rightarrow \int_3^6 e^{-st} \cosh t \, dt = \sinh t e^{-st} \Big|_3^6 - \int_3^6 -s e^{-st} \sinh t \, dt.$$

$$= \sinh t e^{-st} \Big|_3^6 + s \int_3^6 e^{-st} \sinh t \, dt.$$

let $u = e^{-st}$ $du = -s e^{-st} dt$
 $dv = \sinh t \, dt$ $v = \int dv = \int \sinh t \, dt = \cosh t$

$$= \left[\sinh t e^{-st} \right]_3^6 + \left[s \cosh t e^{-st} \Big|_3^6 + s^2 \int_3^6 e^{-st} \cosh t \, dt \right]$$

but $M(t) = \int_3^6 e^{-st} \cosh t \, dt.$

$$\Rightarrow M(t) = \left[\sinh t e^{-st} \right]_3^6 + s \cosh t e^{-st} \Big|_3^6 + s^2 M(t).$$

$$M(t) = \left[e^{-6s} \sinh(6) - e^{-3s} \sinh(3) \right] + s \left[e^{-6s} \cosh(6) - e^{-3s} \cosh(3) \right] + s^2 M(t).$$

$$M(t) - s^2 M(t) = e^{-6s} \sinh(6) - e^{-3s} \sinh(3) + s e^{-6s} \cosh(6) - s e^{-3s} \cosh(3)$$

$$M(t)(1-s^2) = e^{-6s} \sinh(6) - e^{-3s} \sinh(3) + s e^{-6s} \cosh(6) - s e^{-3s} \cosh(3)$$

$$M(t) = \frac{e^{-6s} \sinh(6) - e^{-3s} \sinh(3) + s e^{-6s} \cosh(6) - s e^{-3s} \cosh(3)}{1-s^2}. \quad (ii)$$

and $\int_3^6 t e^{-st} \, dt$, let $u = t$ $du = dt$
 $dv = e^{-st} \, dt$ $v = \int dv = \int e^{-st} \, dt = -\frac{e^{-st}}{s}$

$$\text{then, } \int_3^6 t e^{-st} \, dt = -\frac{t}{s} e^{-st} \Big|_3^6 + \frac{1}{s} \int_3^6 e^{-st} \, dt.$$

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$$\Rightarrow \int_3^6 t e^{-st} dt = \left. -\frac{t}{s} e^{-st} \right|_3^6 - \frac{1}{s^2} e^{-st} \Big|_3^6$$

$$= -\frac{6}{s} e^{-6s} + \frac{3}{s} e^{-3s} - \frac{1}{s^2} e^{-6s} + \frac{1}{s^2} e^{-3s} \quad \text{--- (iii)}$$

Then from equation (i), (ii) and (iii) equation (i) becomes,

$$e^{-3s} \left(\frac{e^{-6s} \sinh(6) - e^{3s} \sinh(3) + s e^{-6s} \cosh(6) - s e^{-3s} \cosh(3)}{1-s^2} + \frac{3e^{-3s} - 6e^{-6s}}{s} + \frac{e^{-3s} - e^{-6s}}{s^2} \right) = \dots \quad \text{--- (iv)}$$

Consider $e^{-6s} \mathcal{L}(7 - \cosh t) = \int_3^6 e^{-6s} e^{-st} (7 - \cosh t) dt$

$$\Rightarrow \int_3^6 e^{-6s} e^{-st} (7 - \cosh t) dt = e^{-6s} \int_3^6 e^{-st} (7 - \cosh t) dt$$

$$W(x) = e^{-6s} \int_3^6 7 e^{-st} dt - \int_3^6 e^{-st} \cosh t dt \quad \dots \text{--- (v)}$$

Let $u = e^{-st}$ $du = -s e^{-st} dt$ hold e^{-6s} to be constant

$$dv = \cosh t dt \quad v = \int dv = \int \cosh t dt = \sinh t$$

$$\Rightarrow \int_3^6 e^{-st} \cosh t dt = \left. e^{-st} \sinh t \right|_3^6 + s \int_3^6 e^{-st} \sinh t dt$$

Then equation (v) becomes

$$= \left. \frac{7}{-s} e^{-st} \right|_3^6 - \left. e^{-st} \sinh t \right|_3^6 - s \int_3^6 e^{-st} \sinh t dt$$

$$\text{let } u = e^{-st} \quad du = -s e^{-st} dt$$

$$dv = \sinh t dt \quad v = \int dv = \int \sinh t dt = \cosh t$$

$$= \left. -\frac{7}{s} e^{-st} \right|_3^6 - \left. e^{-st} \sinh t \right|_3^6 - s \left[\left. e^{-st} \cosh t \right|_3^6 + s \int_3^6 e^{-st} \cosh t dt \right]$$

$$\Rightarrow W(x) = \frac{-7}{s} (e^{-6s} - e^{-3s}) - \left[e^{-6s} \sinh(6) - e^{-3s} \sinh(3) \right] - \left[s e^{-6s} \cosh(6) - s e^{-3s} \cosh(3) \right] + s^2 W(x)$$

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$$\Rightarrow W(z) = \frac{1}{s} (1 - e^{-6s}) - \frac{1}{s} (e^{-6s} - e^{-3s}) - \frac{1}{s} (e^{-6s} - e^{-3s})$$

$$= \frac{1 - e^{-6s} - e^{-6s} + e^{-3s} + e^{-6s} - e^{-3s}}{1 - s^2}$$

$$\text{Therefore } F(s) = \frac{1}{s^2} + e^{-3s} \left[\frac{e^{-6s} - e^{-3s} + e^{-6s} - e^{-3s}}{1 - s^2} \right] - \frac{1}{s} (e^{-6s} - e^{-3s})$$

$$= \frac{3e^{-3s} - 6e^{-6s} - e^{-3s} + e^{-6s}}{s^2} + e^{-6s} \left[\frac{-1}{s} (e^{-6s} - e^{-3s}) \right] - \frac{1}{s} (e^{-6s} - e^{-3s})$$

$$1 - s^2$$